Comparing Truncation Error to PDE Solution Error on Spherical Voronoi Tessellations

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What are we trying to do here?

After we develop and implement a new numerical algorithm, we ask the question “how well does it work?”

Since we are solving nonlinear PDEs, exact solutions are few and are between....and those exact solutions are often not particularly interesting.

Alternatively, we can look at the properties of the discrete differential operators. This is often referred to a truncation error analysis and is sometimes used as a substitute to a PDE solution error analysis.

The point of this work is to look at both PDE solution error and discrete differential operator truncation error to see what we learn.

Definitions

Given a partial differential equation of the form $L(u) = f$,

Assume the discrete approximation to the PDE has the form $\hat{L}(\hat{u}) = \overset{\frown}{f}$.

The $\hat{\ }$ denotes a discrete approximation, the $\frown$ defines a projection of an analytic function to the grid. So,

$\hat{L}$ is the discrete operator

$\hat{u}$ is the discrete solution to the PDE

$\overset{\frown}{f}$ is the analytic RHS projected onto the grid.

Truncation Error Analysis: $\hat{L}(\overset{\frown}{u}) - \overset{\frown}{f} = \hat{\tau}$

Solution Error Analysis: $\hat{L}^{-1}(\overset{\frown}{f}) - \overset{\frown}{u} = \hat{\sigma}$

Truncation Error Analysis

Given a partial differential equation of the form $L(u) = f$,

assume we have a continuous solution to this system; so we know u and f.

A truncation error analysis is carried out as follows:

1) choose a grid

2) project u and f onto that grid to form $\frown{u}$ and $\frown{f}$.

3) apply $\hat{L}$ to $\frown{u}$; compute $\hat{f}$

4) compute $\hat{f} - \frown{f}$, call this $\hat{\tau}$ (the truncation error).

Repeat this process for a sequence of grids of increasing resolution. Does $\hat{\tau}$ decrease with decreasing grid size? To answer this quantitatively, we much chose norms.

$$\|\hat{\tau}\|_2 = \left[\frac{1}{A_T}\sum_{i=1}^{N} A(i)[\hat{\tau}(i)]^2\right]^{\frac{1}{2}} \qquad \|\hat{\tau}\|_\infty = max[|\hat{\tau}(i)|]\Big|_{i=1}^{N}$$

If the norms of $\hat{\tau}$ decrease with increasing resolution, the operator is consistent.

Solution Error Analysis

Given a partial differential equation of the form $L(u) = f$,

assume we have a continuous solution to this system; so we know u and f.

A solution error analysis is carried out as follows:

1) choose a grid

2) project u and f onto that grid to form $\frown{u}$ and $\frown{f}$.

3) invert $\hat{L}$ and apply to $\frown{f}$; solve for $\hat{u}$

4) Compute $\hat{u} - \frown{u}$, call this $\hat{\sigma}$ (the solution error).

Repeat this process for a sequence of grids of increasing resolution. Does $\hat{\sigma}$ decrease with decreasing grid size? To answer this quantitatively, we much chose norms.

$$\|\hat{\sigma}\|_2 = \left[\frac{1}{A_T}\sum_{i=1}^{N} A(i)[\hat{\sigma}(i)]^2\right]^{\frac{1}{2}} \qquad \|\hat{\sigma}\|_\infty = max[|\hat{\sigma}(i)|]\Big|_{i=1}^{N}$$

If the norms of $\hat{\sigma}$ decrease with increasing resolution, the operator is convergent.

Relating Truncation and Solution Error

Solution Error Analysis: $\hat{L}^{-1}(\overset{\frown}{f}) - \overset{\frown}{u} = \hat{\sigma}$, so $\overset{\frown}{u} = \hat{L}^{-1}(\overset{\frown}{f}) - \hat{\sigma}$.

Truncation Error Analysis: $\hat{L}(\overset{\frown}{u}) - \overset{\frown}{f} = \hat{\tau}$, so $\boxed{\hat{\sigma} = \hat{L}^{-1}(\hat{\tau})}$

The solution error is equal to the discrete inverse operator applied to the truncation error.

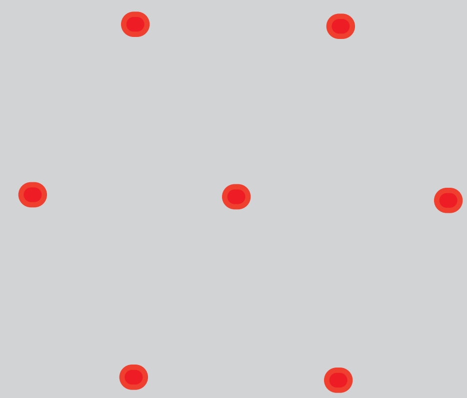
If we assume that $\hat{L}^{-1}$ is a stable approximation to L^{-1}, then we know that $\hat{L}^{-1}$ is bounded. Thus, $\left\|\hat{L}^{-1}\right\| < c$. Taking the norm of $\boxed{}$ gives

$\|\hat{\sigma}\| < \left\|\hat{L}^{-1}\right\| \|(\hat{\tau})\|$ **....... the solution error is bounded from above by the truncation error in terms of convergence rate.**

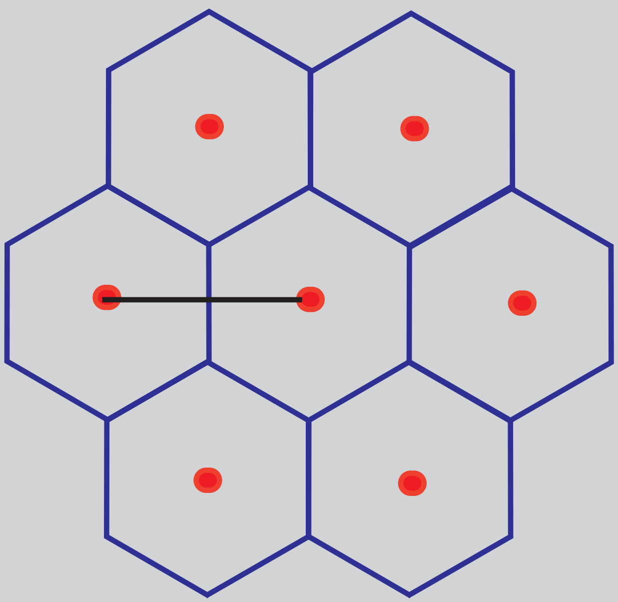
This is the Lax Equivalence Theorem: stability plus consistency guarantees convergence....but what if an operator is not consistent?

Definition of Spherical Voronoi Tesselations

Given the vector positions of a set of points, $\{\tilde{p}\}_{i=1}^{n}$, that lie on the unit sphere, we define for each $\tilde{p}_i$ a corresponding Voronoi region, V_i, as the set of all points on the sphere that lie closer to $\tilde{p}_i$ than $\tilde{p}_j$ for all $j \neq i$. Let each $\tilde{Q}_j$ contain the list of the neighbor locations for each generator location, i.



Properties of SVTs: Every cell wall is an orthogonal bisector of the geodesic connecting the grid points that share that cell wall.

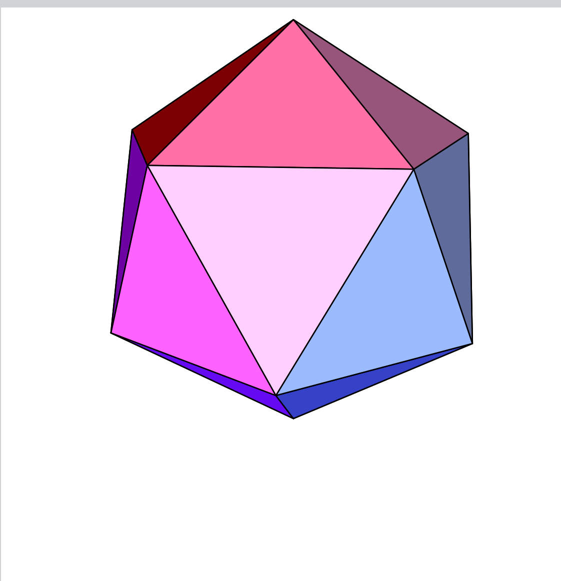


<u>How to chose the generators?</u>

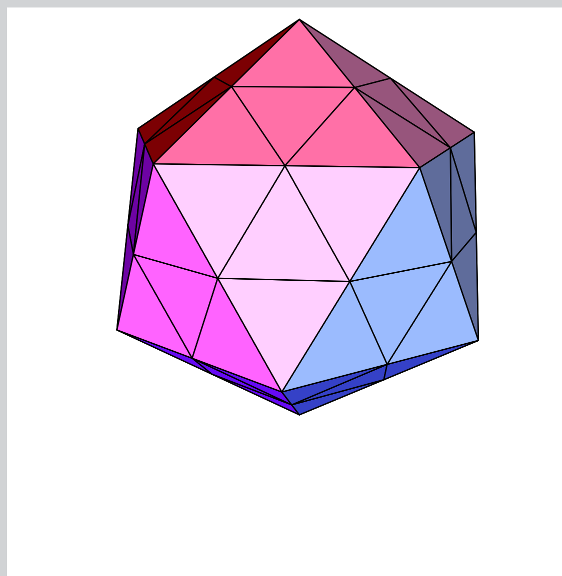
SVTs derived from an inscribed icosahedron

Each vertex will be a grid point (Voronoi region generator)

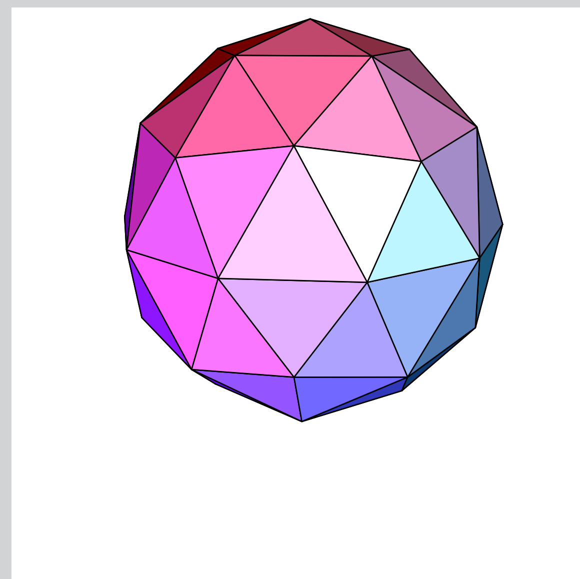
Icosahedron



Bisect each edge and connect



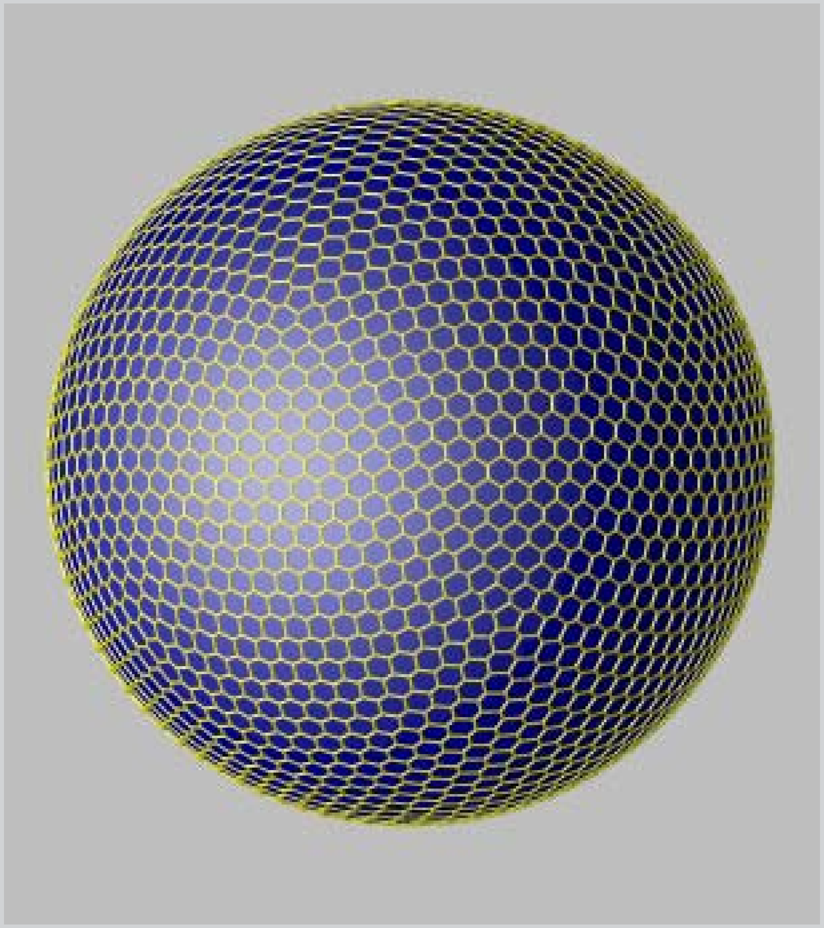
Project new vertices to the sphere



The "unmodified SVT"

Grid Properties

Each Voronoi region is hexagonal in shape, except for twelve regions that are pentagonal. These twelve regions correspond to the vertices of the original icosahedron.

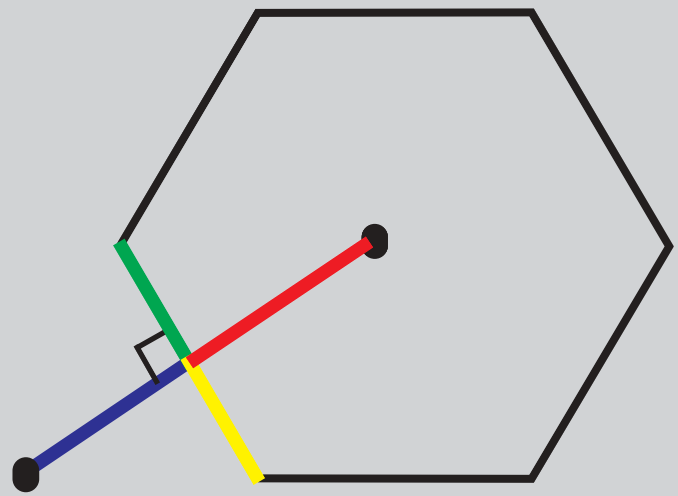


Highly-uniform in horizontal coverage
Highly-uniform in refinement
Highly isotropic
No problematic grid singularity

(The numerical methods we have developed work for any trivalent grid, so let's look at a couple other grids.)

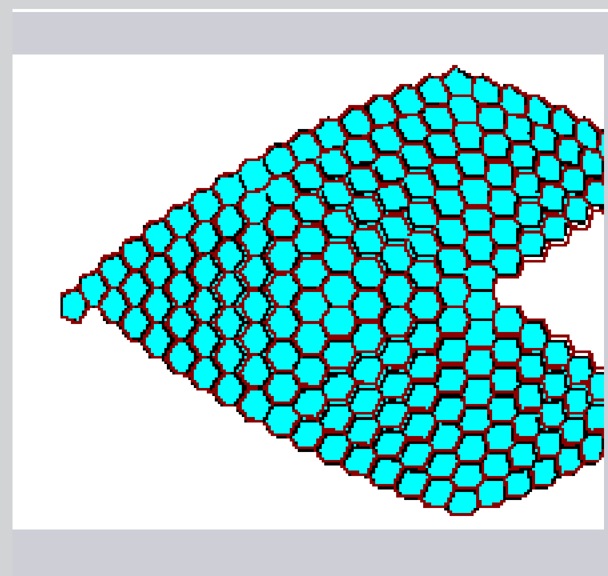
HR95 Grid Optimization Technique

Objective: Modify locations of generating points such that the line segment connecting grid points bisects the cell wall.



Note: A Voronoi grid guarantees that the cell wall segment will bisect the line segment connecting grid points. The converse is not true in general.

The figure to the immediate right shows the unmodified SVT overlaid with the HR95 SVT. The differences are small, but important.

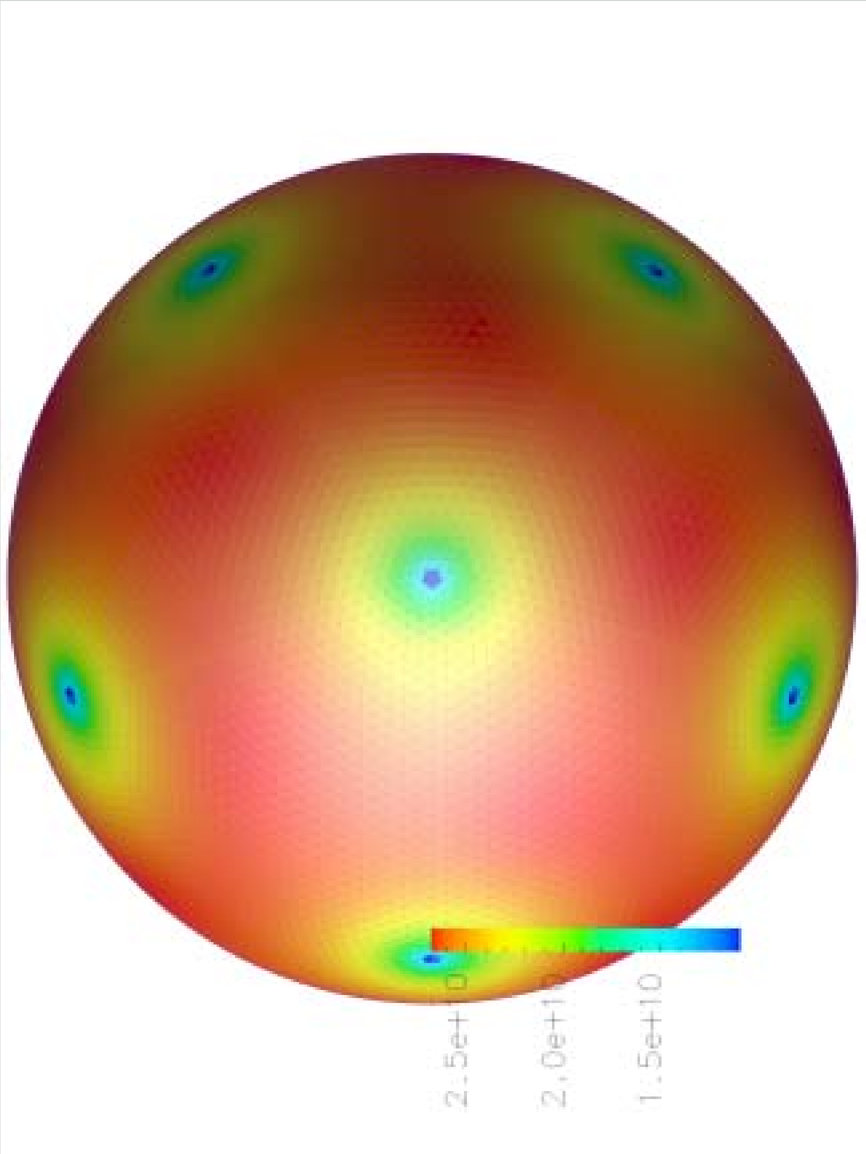


MESQUITE Grid Optimization Technique

Meshes 1 through 4 are created using the condition number, size, smoothness, local area-ratio metrics, respectively. We will focus here on meshes 3 and 4.

A L2 norm is used to create the objective function, meaning that the average mesh quality metric is minimized.

The objective function that is minimized took into account all of the vertices of the mesh simultaneously (as opposed to a series of separate optimization problems) in order to preserve mesh symmetry.

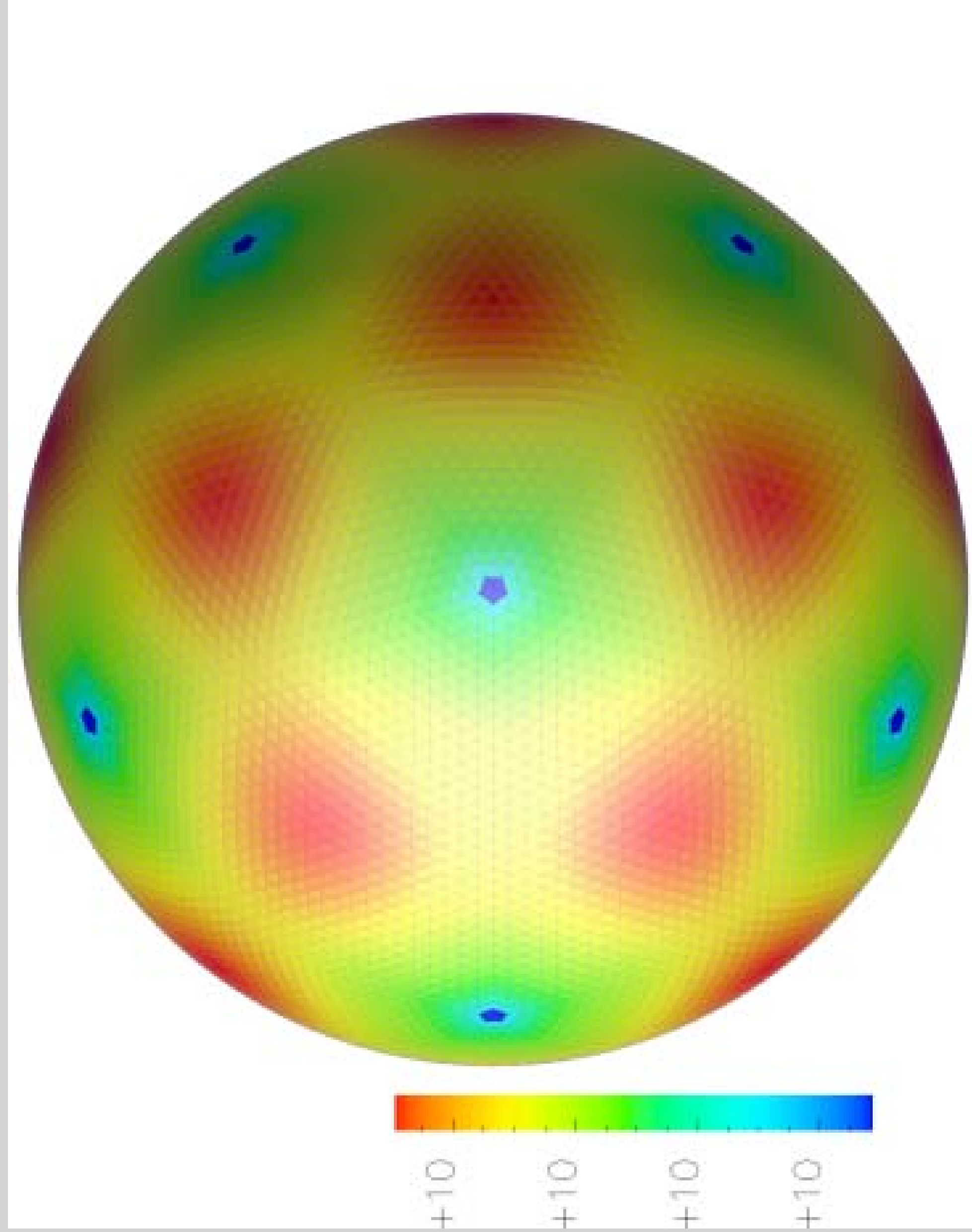


Centroidal Spherical Voronoi Tessellations

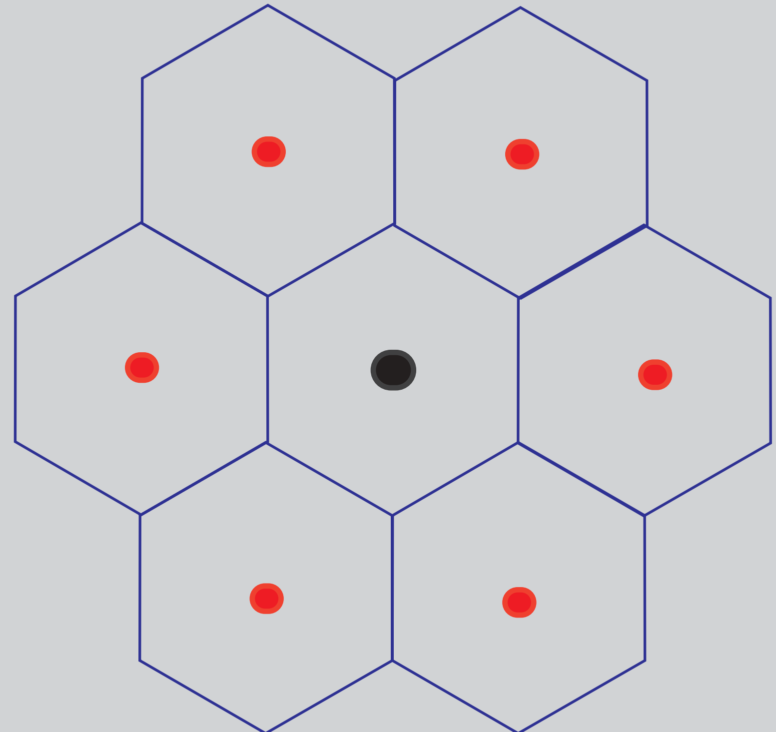
Lloyd’s Algorithm is used to generate the centroidal Voronoi tessellations.

Begin with the unmodified SVT. Find the centroid of each Voronoi region. Move the generator to the centroid of its region. Recompute the Voronoi region and iterate.

Du, Faber, and Gunzburger (SIAM Review, 1999) explore these centroidal Voronoi tessellations on the plane.



Defining global and local uniformity metrics



Local Uniformity:

$$(LocalU)_i = \frac{min[|\tilde{p}_i - \tilde{Q}_j|]\Big|_{j=1}^{\#\text{ neighbors}}}{max[|\tilde{p}_i - \tilde{Q}_j|]\Big|_{j=1}^{\#\text{ neighbors}}}$$

Global Uniformity:

$$GobalU = \frac{min\left[min[|\tilde{p}_i - \tilde{Q}_j|]\Big|_{j=1}^{\#\text{ neighbors}}\right]\Bigg|_{i=1}^{N}}{max\left[max[|\tilde{p}_i - \tilde{Q}_j|]\Big|_{j=1}^{\#\text{ neighbors}}\right]\Bigg|_{i=1}^{N}}$$

A Comparison of the Tessellations

In terms of global uniformity....

Resolution	GlobalU unmod	GlobalU HR1995	GlobalU TSTT03	GlobalU TSTT04	GlobalU Centroid
10242	0.837	0.788	.819	0.589	0.784
40962	0.834	0.787	.811	0.545	0.772
163842	0.834	----	.805	0.460	0.741

And in terms of local uniformity....

resolution	LocalU unmod	LocalU HR1995	LocalU TSTT03	LocalU TSTT04	LocalU Centroid
10242	0.887	0.884	.902	0.956	0.920
40962	0.893	0.885	.903	0.967	0.916
163842	0.896	----	.902	0.969	0.917

SVT Evaluation

Assume a discrete PDE of the form $\hat{L}(\hat{u}) = \widehat{f}$.

$\hat{L}$ is the discrete operator

$\hat{u}$ is the discrete solution to the PDE

$\widehat{f}$ is the right-hand side forcing evaluated at grid locations.

Truncation Error Analysis: $\hat{L}(\widehat{u}) - \widehat{f} = \hat{\tau}$

Solution Error Analysis: $\hat{L}^{-1}(\widehat{f}) - \widehat{u} = \hat{\sigma}$

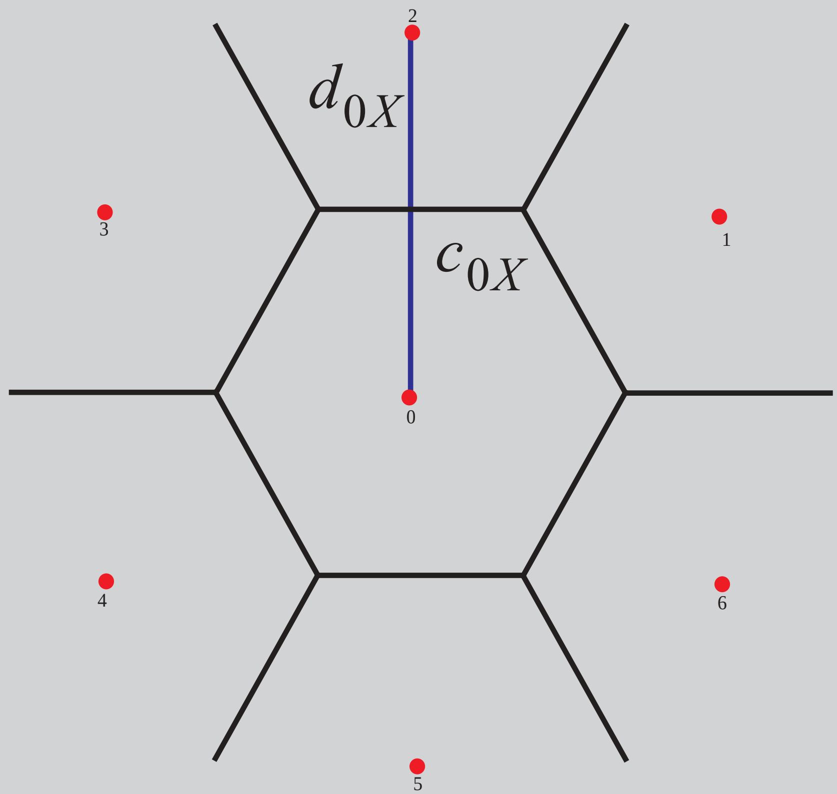
Let $L = \nabla^2$ and look at two exact solutions where

$u = \sin\phi$ [solution#1]

$u = \sin(3\lambda)[\cos(3\phi)]^4$ [solution#2]

Discrete Laplacian

The discrete Laplacian operator based at the cell labeled “0” is a function of cells 0 through 6.



Let d_{0X} denote the distance between cell 0 and cell X.

Let c_{0X} denote the length of the cell shared by 0 and X.

The equation for the Laplacian is then given by

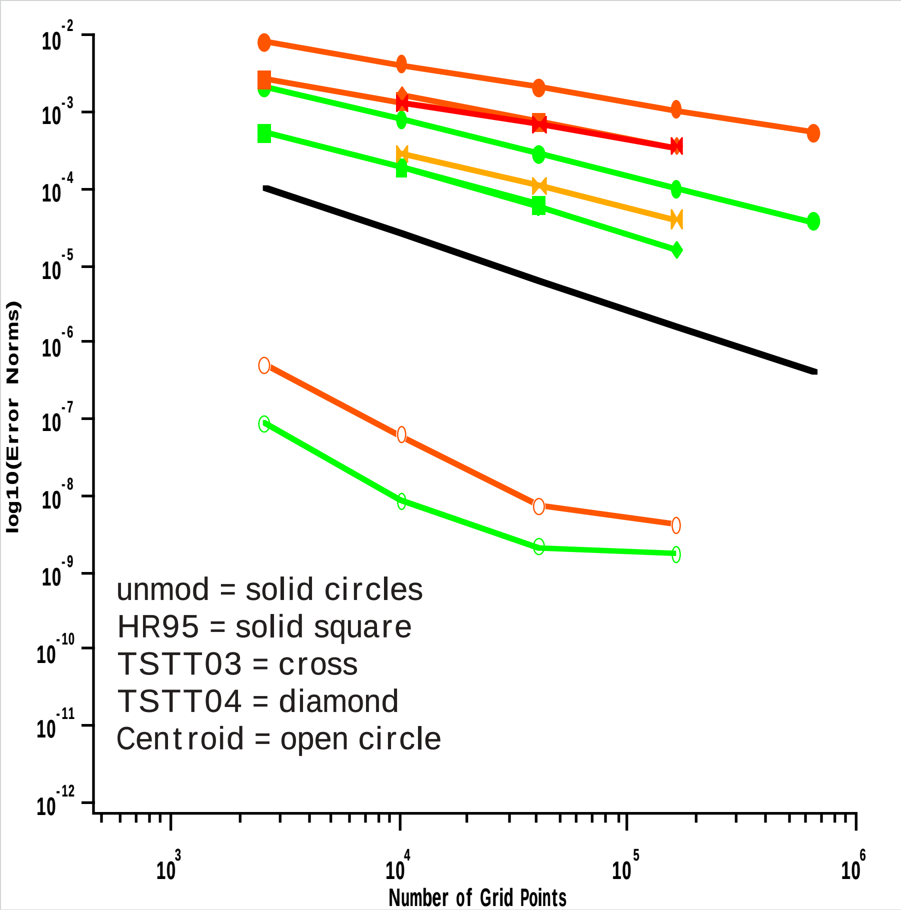
$$L(q_i) = \left[\sum_{j=1}^{\#\text{ neighbors}} \frac{c_{ij}}{A_i d_{ij}} q_j\right] - \frac{e_i}{A_i} q_i$$

Note: This Laplacian is derived as the divergence of the gradient. Operator is valid for all trivalent grids.

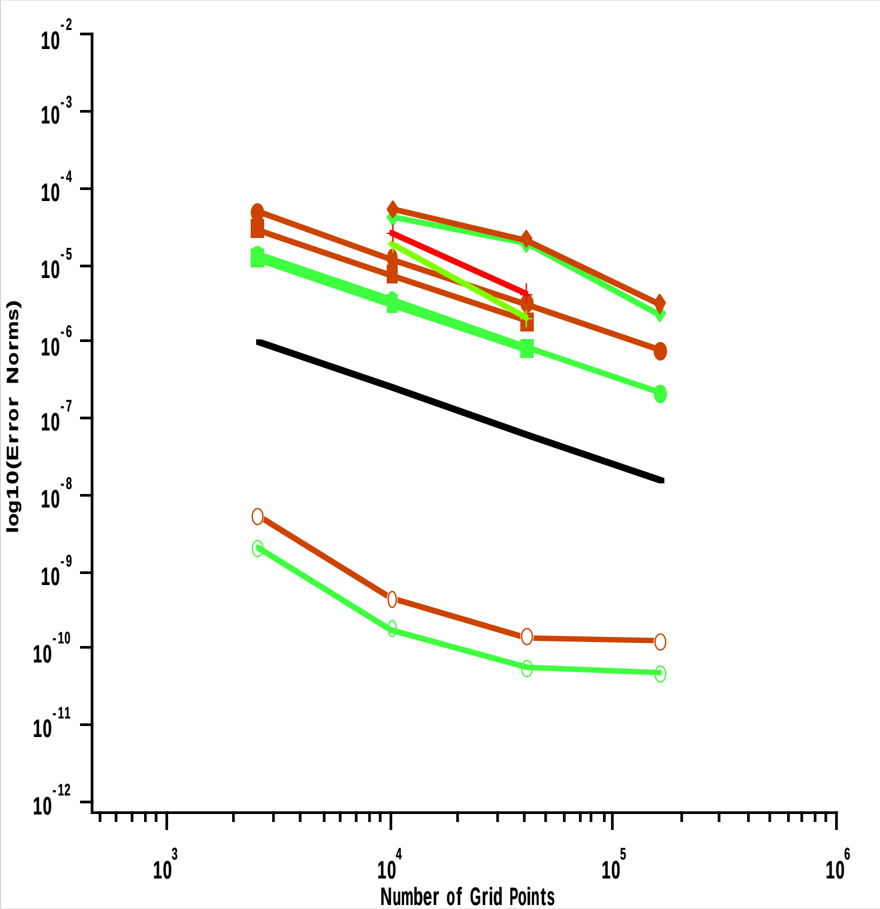
Truncation and Solution Error Results

Solution #1: $u = \sin\phi$

Truncation Error
L2 Norm in Green, Linf Norm in Red
Black Line indicates -2 convergence



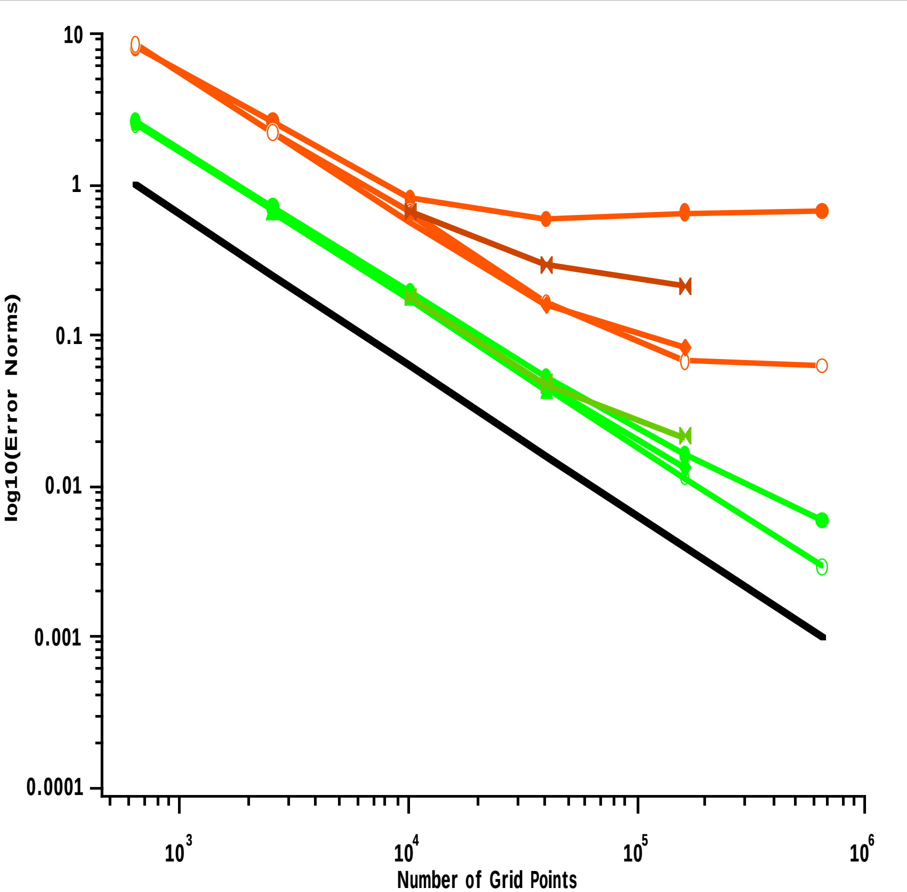
Solution Error
L2 Norm in Green, Linf Norm in Red
Black Line indicates -2 convergence



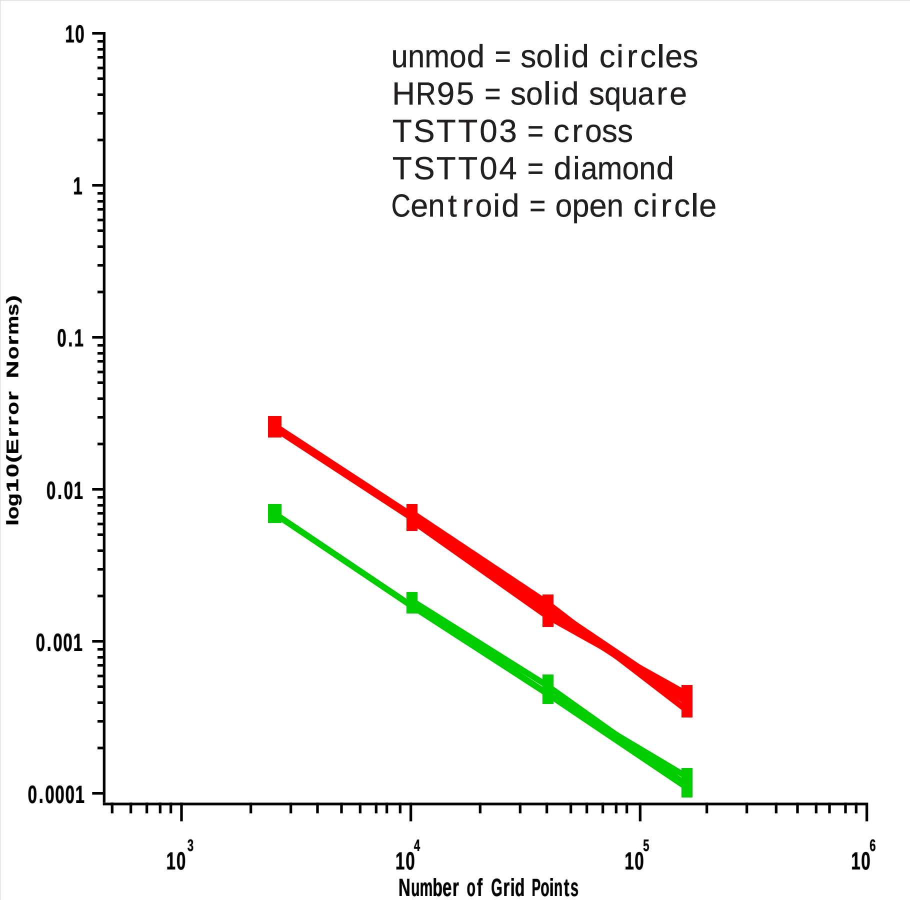
Truncation and Solution Error Results

Solution#2: $u = \sin(3\lambda)[\cos(3\phi)]^4$

Truncation Error
L2 Norm in Green, Linf Norm in Red
Black Line indicates -2 convergence



Solution Error
L2 Norm in Green, Linf Norm in Red



The "poor" truncation error results are not reflected in the solution error results.

What is going on here?

Recall that the Lax Equivalence Theorem says that stability plus consistency is **sufficient** for convergence.

The key here is **sufficient**, as opposed to **necessary**.

Recall, $\|\hat{\sigma}\| < \|\hat{L}^{-1}\| \|(\hat{\tau})\|$. So when $L = \nabla^2$, the solution error is a smoothing of the truncation error. The smoothing is sufficient in the case to increase the order of accuracy of the solution.

This phenomenon is called supra-convergence (Kreiss 1986).

What have I learned here?

In agreement with previous findings, truncation error provides an upper bound in terms of convergence rate.

Optimizing SVTs based on truncation error alone is probably not appropriate.

Regarding the Linf norm, we see O(1) truncation error reduced to $O(h^2)$ solution error.

The failure of the Linf norm for discrete Laplacian operators appears to be common (if not ubiquitous) on Delauny triangulation / Voronoi diagrams.

Super-convergence of the lowest spherical harmonic is also found by Frederickson using his polynomial reconstruction method.